11-7 The Binomial Theorem

Main Ideas

- Use Pascal's triangle to expand powers of binomials.
- Use the Binomial Theorem to expand powers of binomials.

New Vocabulary

Pascal's triangle
Binomial Theorem
factorial

GET READY for the Lesson

According to the U.S. Census Bureau, ten percent of families have three or more children. If a family has four children, there are six sequences of births of boys and girls that result in two boys and two girls. These sequences are listed below.

BBGG BGBG BGGB GBBG GBGB GGBB

Pascal's Triangle You can use the coefficients in powers of binomials to count the number of possible sequences in situations such as the one above. Expand a few powers of the binomial $b + g$.

$$\begin{aligned}
(b+g)^0 &= 1b^0g^0\\
(b+g)^1 &= 1b^1g^0 + 1b^0g^1\\
(b+g)^2 &= 1b^2g^0 + 2b^1g^1 + 1b^0g^2\\
(b+g)^3 &= 1b^3g^0 + 3b^2g^1 + 3b^1g^2 + 1b^0g^3\\
(b+g)^4 &= 1b^4g^0 + 4b^3g^1 + 6b^2g^2 + 4b^1g^3 + 1b^0g^4
\end{aligned}$$

The coefficient 4 of the b^1g^3 term in the expansion of $(b + g)^4$ gives the number of sequences of births that result in one boy and three girls.

Here are some patterns in any binomial expansion of the form $(a + b)^n$.

1. There are $n + 1$ terms.
2. The exponent n of $(a + b)^n$ is the exponent of a in the first term and the exponent of b in the last term.
3. In successive terms, the exponent of a decreases by one, and the exponent of b increases by one.
4. The sum of the exponents in each term is n.
5. The coefficients are symmetric. They increase at the beginning of the expansion and decrease at the end.

The coefficients form a pattern that is often displayed in a triangular formation. This is known as **Pascal's triangle**. Notice that each row begins and ends with 1. Each coefficient is the sum of the two coefficients above it in the previous row.

$$\begin{array}{lccccccccccc}
(a+b)^0 & & & & & & 1 & & & & & \\
(a+b)^1 & & & & & 1 & & 1 & & & & \\
(a+b)^2 & & & & 1 & & 2 & & 1 & & & \\
(a+b)^3 & & & 1 & & 3 & & 3 & & 1 & & \\
(a+b)^4 & & 1 & & 4 & & 6 & & 4 & & 1 & \\
(a+b)^5 & 1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

Real-World Link
Although he did not discover it, Pascal's triangle is named for the French mathematician Blaise Pascal (1623–1662).

EXAMPLE Use Pascal's Triangle

1 Expand $(x + y)^7$.

Write two more rows of Pascal's triangle. Then use the patterns of a binomial expansion and the coefficients to write the expansion.

$$\begin{array}{ccccccccccccccc} & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\ 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 \end{array}$$

$$\begin{aligned}(x + y)^7 &= 1x^7y^0 + 7x^6y^1 + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7x^1y^6 + 1x^0y^7 \\ &= x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7\end{aligned}$$

CHECK Your Progress

1. Expand $(c + d)^8$.

Study Tip

Terms

The expansion of a binomial to the nth power has $n + 1$ terms. For example, $(a - b)^6$ has 7 terms.

The Binomial Theorem Another way to show the coefficients in a binomial expansion is to write them in terms of the previous coefficients.

$$\begin{array}{llllll} (a+b)^0 & 1 \\ (a+b)^1 & 1 & \frac{1}{1} \\ (a+b)^2 & 1 & \frac{2}{1} & \frac{2 \cdot 1}{1 \cdot 2} \\ (a+b)^3 & 1 & \frac{3}{1} & \frac{3 \cdot 2}{1 \cdot 2} & \frac{3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3} \\ (a+b)^4 & 1 & \frac{4}{1} & \frac{4 \cdot 3}{1 \cdot 2} & \frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3} & \frac{4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4} \end{array}$$

Eliminate common factors that are shown in color.

This pattern is summarized in the **Binomial Theorem**.

KEY CONCEPT *Binomial Theorem*

If n is a nonnegative integer, then $(a + b)^n = 1a^nb^0 + \frac{n}{1}a^{n-1}b^1 + \frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^{n-3}b^3 + \ldots + 1a^0b^n$.

EXAMPLE Use the Binomial Theorem

2 Expand $(a - b)^6$.

Use the sequence $1, \frac{6}{1}, \frac{6 \cdot 5}{1 \cdot 2}, \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3}$ to find the coefficients for the first four terms. Then use symmetry to find the remaining coefficients.

$$\begin{aligned}(a - b)^6 &= 1a^6(-b)^0 + \frac{6}{1}a^5(-b)^1 + \frac{6 \cdot 5}{1 \cdot 2}a^4(-b)^2 + \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3}a^3(-b)^3 + \ldots \\ &\quad + 1a^0(-b)^6 \\ &= a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6\end{aligned}$$

Study Tip

Coefficients

Notice that in terms having the same coefficients, the exponents are reversed, as in $15a^4b^2$ and $15a^2b^4$.

CHECK Your Progress

2. Expand $(w + z)^5$.

Study Tip

Graphing Calculators

On a TI-83/84 Plus, the factorial symbol, !, is located on the MATH PRB menu.

The factors in the coefficients of binomial expansions involve special products called **factorials**. For example, the product $4 \cdot 3 \cdot 2 \cdot 1$ is written 4! and is read *4 factorial*. In general, if n is a positive integer, then $n! = n(n-1)(n-2)(n-3) \ldots 2 \cdot 1$. *By definition, 0! = 1.*

EXAMPLE 3 Factorials

Evaluate $\frac{8!}{3!5!}$.

$$\frac{8!}{3!5!} = \frac{8 \cdot 7 \cdot 6 \cdot \cancel{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}^{1}}{3 \cdot 2 \cdot 1 \cdot \cancel{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}_{1}}$$

Note that $8! = 8 \cdot 7 \cdot 6 \cdot 5!$, so $\frac{8!}{3!5!} = \frac{8 \cdot 7 \cdot 6 \cdot 5!}{3!5!}$ or $\frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1}$.

$$= \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} \text{ or } 56$$

CHECK Your Progress

3. Evaluate $\frac{12!}{8!4!}$.

Study Tip

Missing Steps

If you don't understand a step like $\frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} = \frac{6!}{3!3!}$, work it out on a piece of scrap paper.

$$\frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} = \frac{6 \cdot 5 \cdot 4 \cdot 3!}{1 \cdot 2 \cdot 3 \cdot 3!} = \frac{6!}{3!3!}$$

The Binomial Theorem can be written in factorial notation and in sigma notation.

KEY CONCEPT *Binomial Theorem, Factorial Form*

$$(a+b)^n = \frac{n!}{n!0!}a^nb^0 + \frac{n!}{(n-1)!1!}a^{n-1}b^1 + \frac{n!}{(n-2)!2!}a^{n-2}b^2 + \ldots + \frac{n!}{0!n!}a^0b^n$$

$$= \sum_{k=0}^{n} \frac{n!}{(n-k)!k!}a^{n-k}b^k$$

EXAMPLE 4 Use a Factorial Form of the Binomial Theorem

Expand $(2x+y)^5$.

$$(2x+y)^5 = \sum_{k=0}^{5} \frac{5!}{(5-k)!k!}(2x)^{5-k}y^k$$ Binomial Theorem, factorial form

$$= \frac{5!}{5!0!}(2x)^5y^0 + \frac{5!}{4!1!}(2x)^4y^1 + \frac{5!}{3!2!}(2x)^3y^2 + \frac{5!}{2!3!}(2x)^2y^3 + \frac{5!}{1!4!}(2x)^1y^4 + \frac{5!}{0!5!}(2x)^0y^5$$ Let $k = 0, 1, 2, 3, 4,$ and 5.

$$= \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 1}(2x)^5 + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 1}(2x)^4y + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 2 \cdot 1}(2x)^3y^2 + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 3 \cdot 2 \cdot 1}(2x)^2y^3 + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 4 \cdot 3 \cdot 2 \cdot 1}(2x)y^4 + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}y^5$$

$$= 32x^5 + 80x^4y + 80x^3y^2 + 40x^2y^3 + 10xy^4 + y^5$$ Simplify.

CHECK Your Progress

4. Expand $(q-3r)^4$.

Sometimes you need to know only a particular term of a binomial expansion. Note that when the Binomial Theorem is written in sigma notation, $k = 0$ for the first term, $k = 1$ for the second term, and so on. In general, the value of k is always one less than the number of the term you are finding.

EXAMPLE Find a Particular Term

5 **Find the fifth term in the expansion of $(p + q)^{10}$.**

First, use the Binomial Theorem to write the expansion in sigma notation.

$$(p + q)^{10} = \sum_{k=0}^{10} \frac{10!}{(10 - k)!k!} p^{10 - k} q^k$$

In the fifth term, $k = 4$.

$$\frac{10!}{(10 - k)!k!} p^{10 - k} q^k = \frac{10!}{(10 - 4)!4!} p^{10 - 4} q^4 \quad k = 4$$

$$= \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} p^6 q^4 \quad \frac{10!}{6!4!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6!}{6!4!} \text{ or } \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1}$$

$$= 210p^6q^4 \quad \text{Simplify.}$$

CHECK Your Progress

5. Find the eighth term in the expansion of $(x - y)^{12}$.

Online Personal Tutor at algebra2.com

CHECK Your Understanding

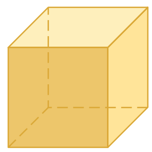
Examples 1, 2, 4 (pp. 665, 666)

Expand each power.

1. $(p + q)^5$ **2.** $(t + 2)^6$ **3.** $(x - 3y)^4$

Examples 2, 4 (pp. 665, 666)

4. GEOMETRY Write an expanded expression for the volume of the cube at the right.



3x + 2 cm

Example 3 (p. 666)

Evaluate each expression.

5. $8!$ **6.** $10!$

7. $\frac{13!}{9!}$ **8.** $\frac{12!}{2!10!}$

Example 5 (p. 667)

Find the indicated term of each expansion.

9. fourth term of $(a + b)^8$ **10.** fifth term of $(2a + 3b)^{10}$

Exercises

HOMEWORK HELP

For Exercises	See Examples
11–16	1, 2, 4
17–20	3
21–26	5

Expand each power.

11. $(a - b)^3$ **12.** $(m + n)^4$ **13.** $(r + s)^8$

14. $(m - a)^5$ **15.** $(x + 3)^5$ **16.** $(a - 2)^4$

Evaluate each expression.

17. $9!$ **18.** $13!$ **19.** $\frac{9!}{7!}$ **20.** $\frac{7!}{4!}$

Find the indicated term of each expansion.

21. sixth term of $(x - y)^9$

22. seventh term of $(x + y)^{12}$

23. fourth term of $(x + 2)^7$

24. fifth term of $(a - 3)^8$

25. SCHOOL Mr. Hopkins is giving a five-question true-false quiz. How many ways could a student answer the questions with three trues and two falses?

Cross-Curricular Project

Math Online Pascal's triangle displays many patterns. Visit algebra2.com to continue work on your project.

26. INTRAMURALS Ofelia is taking ten shots in the intramural free-throw shooting competition. How many sequences of makes and misses are there that result in her making eight shots and missing two?

Expand each power.

27. $(2b - x)^4$

28. $(2a + b)^6$

29. $(3x - 2y)^5$

30. $(3x + 2y)^4$

31. $\left(\frac{a}{2} + 2\right)^5$

32. $\left(3 + \frac{m}{3}\right)^5$

Evaluate each expression.

33. $\frac{12!}{8!4!}$

34. $\frac{14!}{5!9!}$

Find the indicated term of each expansion.

35. fifth term of $(2a + 3b)^{10}$

36. fourth term of $(2x + 3y)^9$

37. fourth term of $\left(x + \frac{1}{3}\right)^7$

38. sixth term of $\left(x - \frac{1}{2}\right)^{10}$

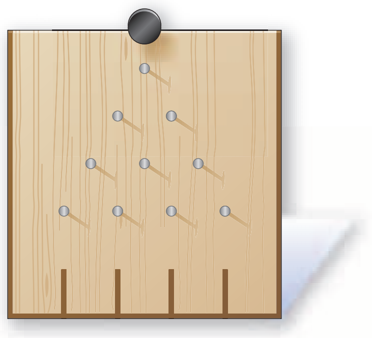
39. GENETICS The color of a particular flower may be either red, white, or pink. If the flower has two red alleles R, the flower is red. If the flower has two white alleles w, the flower is white. If the flower has one allele of each color, the flower will be pink. In a lab, two pink flowers are mated and eventually produce 1000 offspring. How many of the 1000 offspring will be pink?

EXTRA PRACTICE

See pages 916, 936.

Math Online Self-Check Quiz at algebra2.com

40. GAMES The diagram shows the board for a game in which disks are dropped down a chute. A pattern of nails and dividers causes the disks to take various paths to the sections at the bottom. How many paths through the board lead to each bottom section?



H.O.T. Problems

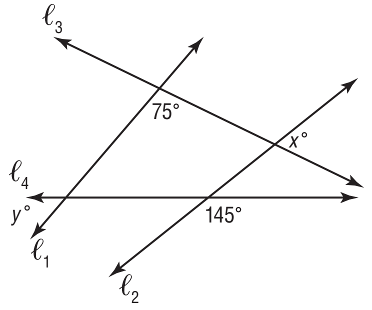
41. OPEN ENDED Write a power of a binomial for which the first term of the expansion is $625x^4$.

42. CHALLENGE Explain why $\frac{12!}{7!5!} + \frac{12!}{6!6!} = \frac{13!}{7!6!}$ without finding the value of any of the expressions.

43. *Writing in Math* Use the information on page 664 to explain how the power of a binomial describes the number of boys and girls in a family. Include the expansion of $(b + g)^5$ and an explanation of what it tells you about sequences of births of boys and girls in families with five children.

STANDARDIZED TEST PRACTICE

44. **ACT/SAT** If four lines intersect as shown, what is the value of $x + y$?



A 70 **B** 115 **C** 140 **D** 220

45. **REVIEW** $(2x - 2)^4 =$

F $16x^4 + 64x^3 - 96x^2 - 64x + 16$

G $16x^4 - 32x^3 - 192x^2 - 64x + 16$

H $16x^4 - 64x^3 + 96x^2 - 64x + 16$

J $16x^4 + 32x^3 - 192x^2 - 64x + 16$

Spiral Review

Find the first five terms of each sequence. (Lesson 11-6)

46. $a_1 = 7, a_{n+1} = a_n - 2$

47. $a_1 = 3, a_{n+1} = 2a_n - 1$

48. **MINIATURE GOLF** A wooden pole swings back and forth over the cup on a miniature golf hole. One player pulls the pole to the side and lets it go. Then it follows a swing pattern of 25 centimeters, 20 centimeters, 16 centimeters, and so on until it comes to rest. What is the total distance the pole swings before coming to rest? (Lesson 11-5)

Without writing the equation in standard form, state whether the graph of each equation is a *parabola*, *circle*, *ellipse*, or *hyperbola*. (Lesson 10-6)

49. $x^2 - 6x - y^2 - 3 = 0$

50. $4y - x + y^2 = 1$

Express each logarithm in terms of common logarithms. Then approximate its value to four decimal places. (Lesson 9-4)

51. $\log_2 5$

52. $\log_3 10$

53. $\log_5 8$

Determine any vertical asymptotes and holes in the graph of each rational function. (Lesson 8-3)

54. $f(x) = \dfrac{1}{x^2 + 5x + 6}$

55. $f(x) = \dfrac{x + 2}{x^2 + 3x - 4}$

56. $f(x) = \dfrac{x^2 + 4x + 3}{x + 3}$

GET READY for the Next Lesson

PREREQUISITE SKILL **State whether each statement is *true* or *false* when $n = 1$. Explain.** (Lesson 1-1)

57. $1 = \dfrac{n(n + 1)}{2}$

58. $1 = \dfrac{(n + 1)(2n + 1)}{2}$

59. $1 = \dfrac{n^2 (n + 1)^2}{4}$

60. $3^n - 1$ is even.

61. $7^n - 3^n$ is divisible by 4.

62. $2^n - 1$ is prime.